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The Generalized Quasilinearization Method for Partial Differential Equations of First Order

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This paper applies the method of generalized quasilinearization to investigating the monotone iterative technique for partial differential equations of first order. The rate of convergence is also discussed.

1. INTRODUCTION

The method of quasilinearization has been systematically used by Bellman [1, 2] in investigating the approximate solutions to nonlinear differential equations. Recently, Lakshmikantham *et al*. [5, 7–11] generalized the method of quasilinearization by weakening the convex condition which is required in Bellman's work. In this paper, we will extend their methods to discuss the monotone iterative technique for partial differential equations of first order.

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2. MAIN RESULTS

In this paper, we follow the notations in [6] and discuss the initial value problem (IVP) for first order partial differential equation

$$u_t + f(t, x)u_x = g(t, x, u), \qquad u(0, x) = \phi(x), \tag{1}$$

where $f \in C[\Omega, R^n]$, $g \in C[\Omega \times R, R]$, $\Omega = [(t, x): 0 \le t \le T, a \le x \le b, a, b, x \in R^n]$, $fu_x = \sum_{i=1}^n f_i(t, x)u_{x_i}$, and $\phi \in C^1[[a, b], R]$.

We need the following two theorems from [5].

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THEOREM 1 [6, Theorem 2.1]. *Assume that*

(A_0) $\alpha, \beta \in C^1[\Omega, R]$, $\alpha_t + f(t,x)\alpha_x \le g(t,x,\alpha)$, $\alpha(0,x) \le \phi(x)$, *and* $\beta_t + f(t,x)\beta_x \ge g(t,x,\beta)$, $\beta(0,x) \ge \phi(x)$, *for* $(t,x) \in \Omega$;

(A_1) $f(t,x)$ *is quasimonotone nonincreasing in* x *for each* t *and* $0 \ge f(t,a)$, $0 \le f(t,b)$;

(A_2) $g(t,x,u_1) - g(t,x,u_2) \le L(u_1 - u_2)$ *whenever* $u_1 \ge u_2$ *for some* $L \ge 0$.

Then $\alpha(t,x) \le \beta(t,x)$ *on* Ω.

Remark 1. Examining the proof of Theorem 1 [6, Theorem 2.1], one can find that (A_2) can be slightly weakened as follows

(A'_2) For a given $d > 0$, there exists $L > 0$ such that $g(t,x,u_1) - g(t,x,u_2) \le L(u_1 - u_2)$ whenever $-d < u_2 \le u_1 < d$.

This observation is useful in this paper.

THEOREM 2 [6, THEOREM 2.2]. *Assume that* (A_1) *and* (A_2) *hold. Suppose further that*

(A_3) *for each* $(t_0, x_0) \in \Omega$, *there exists a unique solution* $x(t_0, x_0)$ *of*

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{2}$$

on $0 \le t \le T$, $x(t, t_0, x_0)$ *is continuously differentiable with respect to* (t_0, x_0), *and the relation*

$$\frac{\partial x}{\partial t_0}(t, t_0, x_0) + \frac{\partial x}{\partial x_0}(t, t_0, x_0) f(t_0, x_0) = 0$$

holds;

(A_4) *for each* $x_0 \in [a,b]$ *and* $y_0 \in R$, *there exists a unique solution* $y(t, 0, y_0; x_0)$ *of*

$$y' = g(t, x(t,0,x_0), y), \qquad y(0) = y_0 \tag{3}$$

on $0 \le t \le T$, *where* $x(t,0,x_0)$ *is the unique solution of* (2), *and* $y(t,0,y_0;x_0)$ *is continuously differentiable with respect to* (y_0, x_0).

Then there exists a unique solution $u(t,x)$ *for the problem* (1) *on* Ω.

Now we state and prove our theorem.

THEOREM 3. *Assume that* (A_0), (A_1), *and* (A_3) *hold with* $\alpha \le \beta$ *on* Ω. *Suppose that* $g, \phi \in C^1[\Omega \times R, R]$, *and* $g(t,x,u) + \phi(x,t,u)$ *and* $\phi(t,x,u)$ *are convex in* u *for each* $(t,x) \in \Omega$. *Furthermore, for a given* $d > 0$, *there exist* $M_1, M_2 > 0$ *such that*

$$g_u(t,x,u) - g_u(t,x,v) \le M_1(u - v)^k \tag{4}$$

and

$$\phi_u(t,x,u) - \phi_u(t,x,v) \le M_2(u-v)^k, \tag{5}$$

where $-d \le v \le u \le d$ *and* $k > 0$.

If u is a solution of the IVP (1), then there exist monotone sequences $\alpha_n(t,x)$, $\beta_n(t,x)$ and the function $\rho(t,x)$, $r(t,x)$ such that

$$\alpha \le \alpha_1 \le \cdots \le \alpha_n \le \rho \le u \le r \le \beta_n \le \cdots \le \beta_1 \le \beta$$

on Ω, and

$$\lim_{n\to\infty} \alpha_n(t,x) = \rho(t,x), \qquad \lim_{n\to\infty} \beta_n(t,x) = r(t,x)$$

uniformly on $t \in [0,T]$ for each x.

In addition, if (A_4) is satisfied, then ρ and r are actually solutions of the IVP (1) and

$$\alpha \le \rho = u = r \le \beta$$

on Ω. In this case, the rate of convergence is $k+1$.

Proof. Let $G(t,x,u) = g(t,x,u) + \phi(t,x,u)$, and consider the linear IVP

$$u_t + f(t,x)u_x = g(t,x,\eta) + [G_u(t,x,\eta) - \phi_u(t,x,\xi)](u-\eta), \tag{6}$$

where $\xi, \eta \in C[\Omega, R]$, and $\alpha \le \xi$, $\eta \le \beta$ on Ω.

Since $g, \phi \in C^1[\Omega \times R, R]$, Mean Value Theorem yields $M_3, M_4 > 0$ such that

$$\begin{aligned} -M_3(u-v) \le {} & \{g(t,x,\eta) + [G_u(t,x,\eta) - \phi_u(t,x,\xi)](u-\eta)\} \\ & - \{g(t,x,\eta) + [G_u(t,x,\eta) - \phi_u(t,x,\xi)](v-\eta)\} \\ \le {} & M_4(u-v), \end{aligned} \tag{7}$$

where $u \ge v$.

That the right-hand side of (6) is linear in u and $g, \phi \in C^1[\Omega \times R, R]$ implies that (see [3, pp. 95–99]) (A_4) is satisfied with respect to

$$\begin{aligned} y' = {} & g(t, x(t,0,x_0), \eta(t, x(t,0,x_0))) \\ & + [G_u(t, x(t,0,x_0), \eta(t, x(t,0,x_0))) \\ & \quad - \phi_u(t, x(t,0,x_0), \xi(t, x(t,0,x_0)))] \\ & \times (y - \eta(t, x(t,0,x_0))), \\ y(0) = {} & y_0. \end{aligned} \tag{8}$$

By Theorem 2, there exists unique solution $u(t, x)$ of (6) on Ω for each $\eta, \xi \in C[\Omega, R]$ such that $\alpha \leq \xi, \eta \leq \beta$ on Ω.

Let α_1 and β_1 be the solutions of

$$\begin{aligned} \alpha_{1t} + f(t,x) a_{1x} &= g(t,x,\alpha) \\ &\quad + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\alpha_1 - \alpha), \qquad (9) \\ \alpha_1(0,x) &= \phi(x), \end{aligned}$$

and

$$\begin{aligned} \beta_{1t} + f(t,x) \beta_{1x} &= g(t,x,\beta) \\ &\quad + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \beta), \qquad (10) \\ \beta_1(0,x) &= \phi(x). \end{aligned}$$

We prove $\alpha \leq \alpha_1$ first.

$$\begin{aligned} \alpha_t + f(t,x)\alpha_x &\leq g(t,x,\alpha) \\ &= g(t,x,\alpha) + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\alpha - \alpha), \\ \alpha(0,x) &\leq \phi(x). \qquad (11) \end{aligned}$$

Comparing (9) and (11), an application of Theorem 1 yields $\alpha(t,x) \leq \alpha_1(t,x)$. $\beta(t,x) \geq \beta_1(t,x)$ can be proved similarly.

To prove $\beta_1 \geq \alpha$, we need to note that the convexity of $g + \phi$ and ϕ implies

$$g(t,x,u) \geq g(t,x,v) + \phi(t,x,v) + G_u(t,x,v)(u-v) - \phi(t,x,u) \qquad (12)$$

and

$$\phi(t,x,u) \geq \phi(t,x,v) + \phi_u(t,x,v)(u-v). \qquad (13)$$

Then

$$\begin{aligned} &\beta_{1t} + f(t,x)\beta_{1x} \\ &\quad = g(t,x,\beta) + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \beta) \end{aligned}$$

$$\overset{(12)}{\geq} g(t,x,\alpha) + \phi(t,x,\alpha) + G_u(t,x,\alpha)(\beta - \alpha) - \phi(t,x,\beta)$$
$$+ [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \beta)$$
$$= g(t,x,\alpha) + \phi(t,x,\alpha) - \phi(t,x,\beta) + G_u(t,x,\alpha)(\beta_1 - \alpha)$$
$$- \phi_u(t,x,\beta)(\beta_1 - \beta)$$
$$\overset{(13)}{\geq} g(t,x,\alpha) + \phi_u(t,x,\beta)(\alpha - \beta) + G_u(t,x,\alpha)(\beta_1 - \alpha)$$
$$- \phi_u(t,x,\beta)(\beta_1 - \beta)$$
$$= g(t,x,\alpha) + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \alpha),$$
$$\beta_1(0,x) = \phi(x). \tag{14}$$

Comparing (11) and (14), and applying Theorem 1, we have $\alpha(t,x) \leq \beta_1(t,x)$, and analogously $\alpha_1(t,x) \leq \beta(t,x)$.

Next we are to prove $\alpha_1 \leq \beta_1$.

$$\alpha_{1t} + f(t,x)\alpha_{1x}$$
$$= g(t,x,\alpha) + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\alpha_1 - \alpha),$$
$$\overset{(12)}{\leq} g(t,x,\alpha_1) + \phi(t,x,\alpha_1) - \phi(t,x,\alpha) - \phi_u(t,x,\beta)(\alpha_1 - \alpha)$$
$$\leq g(t,x,\alpha_1) + \phi_u(t,x,\alpha_1)(\alpha_1 - \alpha) - \phi_u(t,x,\beta)(\alpha_1 - \alpha)$$
$$\leq g(t,x,\alpha_1), \qquad \text{(by the convexity of } \phi \text{ and } \alpha_1 \leq \beta),$$
$$\alpha_1(0,x) = \phi(x).$$

On the other hand,

$$\beta_{1t} + f(t,x)\beta_{1x}$$
$$= g(t,x,\beta) + [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \beta),$$
$$\overset{(12)}{\geq} g(t,x,\beta_1) + \phi(t,x,\beta_1) + G_u(t,x,\beta_1)(\beta - \beta_1) - \phi(t,x,\beta)$$
$$+ [G_u(t,x,\alpha) - \phi_u(t,x,\beta)](\beta_1 - \beta),$$
$$\geq g(t,x,\beta_1) + \phi(t,x,\beta_1) - \phi(t,x,\beta) - \phi_u(t,x,\beta)(\beta_1 - \beta)$$
$$\overset{(13)}{\geq} g(t,x,\beta_1),$$
$$\beta_1(0,x) = \phi(x).$$

$g \in C^1[\Omega \times R, R]$ guarantees that g satisfies (A'_2) when $\alpha \leq u_2 \leq u_1 \leq \beta$. Theorem 1 together with Remark 1 yields $\alpha_1 \leq \beta_1$.

Thus we have proved

$$\alpha \le \alpha_1 \le \beta_1 \le \beta. \tag{15}$$

In general, let α_n and β_n be the solutions of

$$\begin{aligned} \alpha_{nt} + f(t,x)\alpha_{nx} &= g(t,x,\alpha_{n-1}) \\ &\quad + [G_u(t,x,\alpha_{n-1}) - \phi_u(t,x,\beta_{n-1})](\alpha_n - \alpha_{n-1}), \\ \alpha_n(0,x) &= \phi(x), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \beta_{nt} + f(t,x)\beta_{nx} &= g(t,x,\beta_{n-1}) \\ &\quad + [G_u(t,x,\alpha_{n-1}) - \phi_u(t,x,\beta_{n-1})](\beta_n - \beta_{n-1}), \\ \beta_n(0,x) &= \phi(x), \end{aligned} \tag{17}$$

where $n \ge 1$, $\alpha_0 = \alpha$, $\beta_0 = \beta$. By a standard induction argument, we can prove

$$\alpha \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta. \tag{18}$$

Since $\alpha_n = y_n(t, 0, \phi(x(0,t,x)); x(0,t,x))$ on Ω where $y_n = y_n(t,0,y_0; x_0)$ is the unique solution of (8) with $\eta = \alpha_{n-1}$, $\xi = \beta_{n-1}$, and $y_0 = \phi(x_0)$. Hence $\alpha_n = y_n(t,0,\phi(x_0); x_0)$. The monotone sequence $\{y_n(t,0,\phi(x_0); x_0)\}$ converges uniformly and monotonically and we suppose that $\lim_{n\to\infty} y_n(t,0,\phi(x_0); x_0) = y(t,0,\phi(x_0); x_0)$ on $0 \le t \le T$. It is easy to see that

$$\begin{gathered} y'(t,0,\phi(x_0); x_0) = g(t, x(t,0,x_0), y(t,0,\phi(x_0); x_0)), \\ y(0) = \phi(x_0), \end{gathered}$$

and we define $\rho(t,x) = y(t,0,\phi(x(0,t,x)); x(0,t,x))$ on Ω. Similarly we can define $r(t,x)$ with respect to the monotone sequence $\{\beta_n(t,x)\}$.

If, in addition, (A_4) holds, then the limit functions, ρ, r are actually solutions of (1), and we have

$$\alpha \le \rho = u = r \le \beta$$

on Ω due to the uniqueness of solution of IVP (1) by Theorem 2. In this case, we can discuss the rate of convergence.

Let $p_{n+1}(t,x) = u(t,x) - \alpha_{n+1}(t,x), q_{n+1}(t,x) = \beta_{n+1}(t,x) - u(t,x)$, where $(t,x) \in \Omega$. For each x, we have

$$\begin{aligned}
p'_{n+1} &= g(t,x,u) - g(t,x,\alpha_n) \\
&\quad - [G_u(t,x,\alpha_n) - \phi_u(t,x,\beta_n)](\alpha_{n+1} - \alpha_n) \\
&= G(t,x,u) - G(t,x,\alpha_n) - \phi(t,x,u) + \phi(t,x,\alpha_n) \\
&\quad - [G_u(t,x,\alpha_n) - \phi_u(t,x,\beta_n)](\alpha_{n+1} - u + u - \alpha_n) \\
&\le G_u(t,x,u)p_n - \phi_u(t,x,\alpha_n)p_n \\
&\quad + [G_u(t,x,\alpha_n) - \phi_u(t,x,\beta_n)]p_{n+1} \\
&\quad - [G_u(t,x,\alpha_n) - \phi_u(t,x,\beta_n)]p_n \\
&= [g_u(t,x,u) - g_u(t,x,\alpha_n)]p_n + [\phi_u(t,x,u) - \phi_u(t,x,\alpha_n)]p_n \\
&\quad + [G_u(t,x,\alpha_n) - \phi_u(t,x,\beta_n)]p_{n+1} \\
&\quad + [\phi_u(t,x,u) - \phi_u(t,x,\alpha_n)]p_n \\
&\quad - [\phi_u(t,x,u) - \phi_u(t,x,\beta_n)]p_n \\
&\le M_1 p_n^{k+1} + M_2 p_n^{k+1} + 3Lp_{n+1} + M_2 p_n^{k+1} + M_2 q_n^k p_n \\
&\le (M_1 + 3M_2)p_n^{k+1} + 3Lp_{n+1} + M_2 q_n^{k+1},
\end{aligned}$$

where $L \ge |g_u(t,x,u)|$ and $L \ge |\phi_u(t,x,u)|$ on $\Omega \times R$. Applying a Gronwall-type inequality [4, Lemma 1.1.1], we get

$$\begin{aligned}
&\max_{(t,x)\in\Omega} |u(t,x) - \alpha_{n+1}(t,x)| \\
&\qquad \le Te^{3LT}\Big\{(M_1 + 3M_2) \max_{(t,x)\in\Omega} |u(t,x) - \alpha_n(t,x)|^{k+1} \\
&\qquad\qquad + M_2 \max_{(t,x)\in\Omega} |u(t,x) - \beta_n(t,x)|^{k+1}\Big\}.
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}
&\max_{(t,x)\in\Omega} |\beta_{n+1}(t,x) - u(t,x)| \\
&\qquad \le Te^{3LT}\Big\{(2M_1 + 3M_2) \max_{(t,x)\in\Omega} |\beta_n(t,x) - u(t,x)|^{k+1} \\
&\qquad\qquad + (M_1 + M_2) \max_{(t,x)\in\Omega} |u(t,x) - \alpha_n(t,x)|^{k+1}\Big\}.
\end{aligned}$$

Remark 2. If $g, \phi \in C^2[\Omega \times R, R]$, then conditions (4) and (5) are satisfied for $k = 1$. Therefore we get the quadratic convergence.

The following corollary will discuss the case where g can be split into a difference of two convex or concave functions, which was discussed extensively in [9].

COROLLARY. *Assume that* (A_0), (A_1), *and* (A_3) *hold with* $\alpha \le \beta$ *on* Ω. *Suppose that*

$$g(t,x,u) = g_1(t,x,u) - g_2(t,x,u),$$

where $g_1, g_2 \in C^1[\Omega \times R, R]$ *are convex functions in* u *for each* $(t,x) \in \Omega$. *Furthermore, we have*

$$g_{iu}(t,x,u) - g_{iu}(t,x,v) \le M_i(u - v)^k, \tag{19}$$

where $i = 1, 2, u \ge v$, *and* $k > 0$.

If u is a solution of the IVP (1), then there exist monotone sequences $\alpha_n(t,x)$, $\beta_n(t,x)$ and the functions $\rho(t,x)$, $r(t,x)$ such that

$$\alpha \le \alpha_1 \le \cdots \le \alpha_n \le \rho \le u \le r \le \beta_n \le \cdots \le \beta_1 \le \beta$$

on Ω, and

$$\lim_{n\to\infty} \alpha_n(t,x) = \rho(t,x), \qquad \lim_{n\to\infty} \beta_n(t,x) = r(t,x)$$

uniformly on $t \in [0,T]$ for each x.

In addition, if (A_4) is satisfied, then ρ and r are actually solutions of the IVP (1) and

$$\alpha \le \rho = u = r \le \beta$$

on Ω. In this case, the rate of convergence is $k + 1$.

Proof. Let $\phi(t,x,u) = g_2(t,x,u)$. Then $g + \phi$ is convex in u for each $(t,x) \in \Omega$. This corollary is reduced to Theorem 3.

Remark 3. The proof of the corollary suggests that the conditions imposed by Theorem 2.2 in [10] and Theorem 3.1 in [9] are equivalent.

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